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On the integrability of Bianchi cosmological models

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Abstract. In this paper, we investigate the problem of the integrability of Bianchi class A cosmological models. This class of systems is reduced to the form of Hamiltonian systems with exponential potential forms.

The dynamics of Bianchi class A models is investigated through the Euler–Lagrange equations and geodesic equations in the Jacobi metric. On this basis, we have come to some general conclusions concerning the evolution of the volume of 3-space of constant time. The general form of this function has been found. It can serve as a controller during numerical calculations of the dynamics of cosmological models. The integrability of cosmological models is also discussed from the points of view of different integrability criteria. We show that the dimension of the phase space of Bianchi class A Hamiltonian systems can be reduced by two. We prove that the vector field of the reduced system is polynomial and that it does not admit any analytic, or even formal first integral.

1. Introduction

We shall investigate the dynamics of the most interesting group of homogeneous Bianchi class A cosmological models described by the natural Lagrangian function

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2} g_{\alpha\beta}\dot{q}^{\alpha}\dot{q}^{\beta} - V(q) = T - V(q) \\ &= \frac{1}{4}\sum_{i=1,i<j}^{3} \frac{\mathrm{d}\ln q^i}{\mathrm{d}t}\frac{\mathrm{d}\ln q^j}{\mathrm{d}t} - \frac{1}{4}\left(2\sum_{i=1,i<j}^{3} n_i n_j q^i q^j - \sum_{i=1}^{3} n_i^2 q_i^2\right) \qquad (1)\end{aligned}$$

where $q_i \approx A_i^2$ $(i = 1, 2, 3)$ are three squared scale factors A_i for diagonal class A Bianchi models; different Bianchi types correspond to different choices of $n_i \in \{-1, 0, 1\}$, $i = 1, 2, 3$; a dot denotes differentiation with respect to the cosmological time t. The logarithmic time τ is related to cosmological time t by

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{(q^1 q^2 q^3)^{1/2}} = \frac{\mathrm{d}t}{\mathrm{Vol}\, M^3}.$$

Bogoyavlensky [1] proved an important property of the system (1), namely, the existence of the monotonic function F of the following form:

$$F = \frac{\mathrm{d}}{\mathrm{d}t}(q^1 q^2 q^3)^{1/6} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Vol}\, M^3)^{1/3} \qquad (2)$$

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such that

$$\frac{\mathrm{d}F}{\mathrm{d}t} \leqslant 0.$$

The function (2) is invariant with respect to the scaling transformations and it has the sense of the speed of change of the average radius of the universe. The function $|F|$ along any solution decreases from infinity to zero in such a way that $F = 0$ is reached at the moment of maximal expansion, and $|F| = \infty$ corresponds to the initial singularity. The existence of the function F allows us to define what we call the early stage of the evolution of the universe as

$$F \gg 1.$$

The importance of this function for numerical integration of B(IX) models has been pointed out in [2]. The authors used the Rauchaudhuri equations to show the property of upper-convexity of the function $(\mathrm{Vol}\, M^3)(t)$ which means that this function does not possess a local minimum (where $F = 0$ and $\dot{F} > 0$), and may possess not more than one maximum. If $F < 0$, the volume $(\mathrm{Vol}\, M^3)(t)$ shrinks; whereas if $F > 0$ it expands. Both processes take place in the same region of the phase space (p, q) but with reverse directions of time. In the phase space (p, q) the function F has the following form:

$$F = \frac{(q^1 q^2 q^3)^{1/6}}{3} p_i q^i \qquad p_i = \frac{\partial L}{\partial \dot{q}^i}$$

and

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{(q^1 q^2 q^3)^{1/6}}{9} \left[(p_i q^i)^2 - 6V \right]$$

where the p_i are the momenta conjugated with the generalized coordinates q^i. In [2], the function F was used to control the quality of numerical integrations of the B(IX) model. In this model, the scale factors oscillate in a neighbourhood of the initial and final singularities. The function $(\mathrm{Vol}\, M^3)(t)$, obviously, does not possess the analogous property [2, 3].

Let us note the recent important results of Cushman and Śniatycki [4] concerning the function F and chaos in the B(IX) system. They proved that the existence of a monotonic function F excludes the possibility of recurrence in the system and, thus, any form of standard deterministic chaos in the system. This illuminates previous negative results and shows that for a study of this system we have to use non-conventional methods.

Several authors tested whether the last model passes the standard Painlevé integrability test (in the form of the ARS algorithm [5]). The first results of Contopoulos *et al* [6] showed that the B(IX) model passes this test. Next, this paper was revised [7], however, without any strict conclusions concerning integrability. It was also stated that this model passes Ziglin's test (see [11, 10, 9]). More careful Painlevé analysis was done by Latifi *et al* in [12]. They showed that the B(IX) model does not pass the so-called perturbative Painlevé test. The authors of this paper suggest the existence of 'some chaotic regimes' in the system. The strongest result in this direction was obtained in [8] where the authors showed the existence of movable critical essential singularities in the B(IX) model.

The above remarks show that the notion of 'chaos' has an unclear status when dynamical systems arising from general relativity and cosmology are studied. Moreover, for the B(IX) model a discrete dynamics defined in [13] that approximates the exact continuous model shows strong ergodic properties; however, this 'chaotic behaviour' seems to be absent (or hidden) in the continuous dynamics. Moreover, the standard criteria for the detection of chaos (Lyapunov characteristic exponents, LCE) are not invariant with respect to the time reparametrization and transformation of phase variables, whereas the existence of first integrals is an invariant property of the system. It is also important to note that the non-zero LCE can be used as an indicator of chaos only when the motion takes place in a compact invariant subset of the phase space, but this is not true for the B(IX) dynamical system. All these facts formed the motivation for us to study the problem of integrability of the models investigated here. The non-integrability of the system is a weaker property than chaos (in the sense of deterministic chaos) but is better described and understood. The authors believe that investigations of non-integrability in the B(IX) models can contribute to a better understanding of chaos in cosmological models.

Here we show that the Bianchi class A Hamiltonian systems are not completely integrable in the Birkhoff sense. This conclusion is weak as the negative answer to the question about algebraic complete integrability of B(IX) (see [14]). In order to obtain a stronger result we reduce the dimension of the phase space by two. We show that the reduced system is polynomial and, most importantly, it does not admit any analytical, or even formal, first integral.

In cosmological models chaos, if properly defined and present, has some hidden character. The basic indicator of chaos in these models, the LCE, depends on the choice of the time parametrization. In the logarithmic time τ, nearby trajectories diverge linearly whereas in other time parametrizations they will diverge exponentially, which is characteristic for chaotic systems. The fact that the rates of separation of nearby trajectories depend on the clock used is obvious. The problem lies in the invariant choices of the time parameter for invariant chaos detection. Such a role is played by the Maupertuis clock (the time parameter s is such that $\mathrm{d}s/\mathrm{d}\tau = 2|E - V|$, where E is the total energy of the system, V is its potential and τ is the mechanical time).

Our point of view is such that the LCE, when used in general relativity, should be defined in an invariant way. Then the results could be interpreted in different time parametrizations. The Bianchi IX model is 'chaotic' in the parameter s (the LCE is positive), but, after transition to the parameter τ, nearby trajectories diverge linearly in the same way as in integrable systems. This phenomenon is called 'hidden chaos'. Let us note that the existence of the first integral of an autonomous system is an invariant property (with respect to time reparametrization and to transformation of phase variables). In general relativity and cosmology, the problem of non-integrability or chaos is not only very subtle but also is strictly connected with the invariant description. One must be very careful in trying to detect integrability in the B(IX) dynamics. The question as to whether chaos in the gauge theory is a physical phenomenon is, generally, an open problem. Recently, a fractal approach seems to give the definite answer to the problem of the existence of chaos in this class of systems. Cornish and Levin [15] demonstrated that the mixmaster universe is indeed chaotic by using coordinate-independent, fractal methods. Unfortunately, they used an approximation of the true dynamics.

2. The dynamics of Bianchi class A models from the Euler–Lagrange equations

The Hamiltonian function for the system (1) has the following form:

$$\mathcal{H} = \tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta + V(q) \tag{3}$$

where

$$p_\alpha = g_{\alpha\beta}\dot{q}^\beta$$

$$g^{\alpha\beta} = 2\begin{pmatrix} -(q^1)^2 & q^1q^2 & q^1q^3 \\ q^2q^1 & -(q^2)^2 & q^2q^3 \\ q^3q^1 & q^3q^2 & -(q^3)^2 \end{pmatrix}$$

$$V(q) = \frac{1}{4}\left(2\sum_{i<j}^{3} n_i n_j q^i q^j - \sum_{i=1}^{3} n_i^2 (q^i)^2\right).$$

Here $V(q)$ is the potential function. The obtained Hamiltonian system is considered only on an invariant set of the phase space defined by the zero level of the Hamiltonian (3), i.e.

$$\mathcal{H} = 0. \tag{4}$$

The Euler–Lagrange equations in time τ have the following form:

$$\frac{\mathrm{d}^2 q^\alpha}{\mathrm{d}\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{\mathrm{d}q^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}q^\gamma}{\mathrm{d}\tau} = g^{\alpha\beta}\frac{\partial V}{\partial q^\beta} \tag{5}$$

where the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ are connected with the metric defined by the kinetic energy

$$T = \tfrac{1}{2} g_{\alpha\beta}\dot{q}^\alpha \dot{q}^\beta = \tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta.$$

After being transformed to a new time parameter s, called the Maupertuis time, equations (5) take the form of geodesic equations for the Jacobi metric

$$\hat{g}_{\alpha\beta} = 2|E - V| g_{\alpha\beta} = 2W g_{\alpha\beta}$$

i.e.

$$\frac{\mathrm{d}^2 q^\alpha}{\mathrm{d}s^2} + \hat{\Gamma}^\alpha_{\beta\gamma}\frac{\mathrm{d}q^\beta}{\mathrm{d}s}\frac{\mathrm{d}q^\gamma}{\mathrm{d}s} = 0 \tag{6}$$

where a hat denotes that the quantities are calculated with respect to the Jacobi metric. The Christoffel symbols calculated from g and $\hat{g}$ metrics are connected by the relations

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} + A^i_{jk} \tag{7}$$

where

$$A^i_{jk} = (\partial_j \Phi)\delta^i_k + (\partial_k \Phi)\delta^i_j - g^{ir}(\partial_r \Phi) g_{jk}$$

$$\Phi = \tfrac{1}{2}\ln 2W.$$

Let us note that the kinetic energy form does not depend on the Bianchi type models characterized by the set $\{n_1, n_2, n_3\}$. The only non-vanishing Christoffel symbols are

$$\Gamma^1_{11} = -\frac{1}{q^1} \qquad \Gamma^2_{22} = -\frac{1}{q^2} \qquad \Gamma^3_{33} = -\frac{1}{q^3}. \tag{8}$$

After substitution of (8), the system (5) takes the form

$$\frac{1}{q^i}\frac{\mathrm{d}^2 q^i}{\mathrm{d}\tau^2} - \left(\frac{\mathrm{d}}{\mathrm{d}\tau}\ln q^i\right)^2 = (n_j)^2 (q^j)^2 + (n_k)^2 (q^k)^2 - (n_i)^2 (q^i)^2 - 2n_j n_k q^j q^k \tag{9}$$

where $\{i, j, k\} \in S_3$, and S_3 denotes the set of even permutations of $\{1, 2, 3\}$.

The change of variables

$$q^i = \mathrm{e}^{Q^i} \qquad i = 1, 2, 3 \tag{10}$$

transforms the above equations to the following form:

$$\frac{\mathrm{d}^2 Q^i}{\mathrm{d}\tau^2} = (n_j)^2 \mathrm{e}^{2Q^j} + (n_k)^2 \mathrm{e}^{2Q^k} - (n_i)^2 \mathrm{e}^{2Q^i} - 2n_j n_k \mathrm{e}^{Q^j + Q^k} \tag{11}$$

where $\{i, j, k\} \in S_3$.

After introducing the new variables Q^i and using the definition (10), the Lagrange system (1) can be transformed to a Hamiltonian one. The Hamilton function for this system takes the following form:

$$\mathcal{H}(p, Q) = 2\sum_{i<j}^{3} p_i p_j - \sum_{i=j}^{3} p_i^2 + \frac{1}{4}\left(2\sum_{i<j}^{3} n_i n_j \mathrm{e}^{Q^i+Q^j} - \sum_{i=j}^{3} n_i^2 \mathrm{e}^{2Q^i}\right)$$
$$= \tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta + V(Q^\alpha) \tag{12}$$

where $p_i = \frac{1}{4}(\dot{Q}^j + \dot{Q}^k)$ for $\{i, j, k\} \in S_3$. The Hamilton function (12) is a special case of the Hamiltonian for the so-called perturbed Toda lattice [16].

The system (11) is satisfied on the Hamiltonian constraint $\mathcal{H} = 0$ which is equivalent to the condition of normalization of the tangent vector to the trajectory $u^i = \mathrm{d}q^i/\mathrm{d}s$, i.e.

$$\|u\|^2 = 2W g_{\alpha\beta} \frac{\mathrm{d}q^\alpha}{\mathrm{d}s} \frac{\mathrm{d}q^\beta}{\mathrm{d}s} = -\operatorname{sgn} V \tag{13}$$

or

$$g_{\alpha\beta} \frac{\mathrm{d}q^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}q^\beta}{\mathrm{d}\tau} = 2V \operatorname{sgn} V.$$

In terms of the variables Q^i, the constraint condition is equivalent to

$$\sum_{i<j}^{3} \frac{\mathrm{d}Q^i}{\mathrm{d}\tau} \frac{\mathrm{d}Q^j}{\mathrm{d}\tau} = -8V. \tag{14}$$

Summing both sides of equations (11), we obtain the following formula:

$$\sum_{i=1}^{3} \frac{\mathrm{d}^2 Q^i}{\mathrm{d}\tau^2} = -4V. \tag{15}$$

Equations (11), after time reparametrization $\tau \to s = s(\tau)$, take the form of geodesic equations. From equations (11) we obtain

$$4W^2 \frac{\mathrm{d}^2 Q^i}{\mathrm{d}s^2} + \frac{\mathrm{d}Q^i}{\mathrm{d}s} \frac{\partial W}{\partial Q^j} \frac{\mathrm{d}Q^j}{\mathrm{d}s} = (n_j)^2 \mathrm{e}^{2Q^j} + (n_k)^2 \mathrm{e}^{2Q^k} - (n_i)^2 \mathrm{e}^{2Q^i} - 2n_j n_k \mathrm{e}^{Q^j+Q^k}$$

where $\{i, j, k\} \in S_3$. The problem of the investigation of Lagrange systems with indefinite kinetic energy form is an open one. The first steps in investigating such systems were taken in [17]. In the terminology of [17] our system is a special case, the so-called non-classical simple mechanical system. As was established there, these systems have the following fundamental property. A trajectory of the system can pass through the set $\partial D = \{q : E - V = 0\}$. During this passage the vector tangent to trajectory changes the cone sector defined by the kinetic energy form: $g_{\alpha\beta}(q_0)\xi^\alpha\xi^\beta = 0$ where $\xi^\alpha = \mathrm{d}q^\alpha/\mathrm{d}s$, $q_0 \in \partial D$. In our case the signature of $g_{\alpha\beta}$ is Lorentzian, i.e. $(-, +, \ldots, +)$ (for details see [17]).

In generic situations ($n_i \neq 0$ for $i = 1, 2, 3$) which include the B(VIII) and B(IX) models (mixmaster models), there are analytical and numerical arguments that the function of sign of the potential for a typical trajectory is an infinite sub-sequence which is a one-sided cut of the following double infinite sequence (see [18]):

$$\operatorname{sgn} V = \{\ldots, +1, 0, -1, 0, -1, \ldots\}. \tag{16}$$

If we assume that the sub-sequence (16) is finite, then our system reaches the state $V = 0$ ($W = 0$ in the general case) which corresponds to the Kasner solutions, a finite number of times. During the Kasner epoch the information about the localization of the point on the interval of normal separation (modulo initial localization) grows e times [16]. If the subsequence (16) is finite it means that after $\bar{n}$ epochs $\bar{n}$ bytes (i.e. a finite number of bytes) of information have been lost, whereas we know that our system is chaotic (the loss of infinite information is required for chaos). Let us note that when the system goes asymptotically to the boundary sets $W = 0$ then it is asymptotically free.

3. Properties of the volume function of the constant time 3-space

Equation (15) implies that

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\ln(\mathrm{Vol}\,M^3) = -2V = 2T. \tag{17}$$

The above relation means that, in a generic case ($\forall\, i\ n_i \neq 0$), there is an infinite number of intervals in which the function $\ln(\mathrm{Vol}\,M^3)$ is subsequently convex up and down. These intervals are separated by an infinite number of inflexion points (which correspond to $V = 0$) in the diagrams of the function $\ln(\mathrm{Vol}\,M^3)(\tau)$ and lie on the lines $\ln(\mathrm{Vol}\,M^3)(\tau) = \pm\tau + C$. The additional information we have about the B(IX) model is that this model has initial and final singularities. In the following paragraphs we shall concentrate on the B(IX) models. From the $(0, 0)$ components of the Einstein equation for the B(IX) case, we obtain the result that the function $\ln(\mathrm{Vol}\,M^3)(\tau)$ cannot possess a local minimum, but it can possess a single maximum.

After integration of both sides of (15) over τ and assuming that in the moment of maximal expansion $\tau = \tau_0$ we obtain

$$\ln(\mathrm{Vol}\,M^3)(\tau) \propto \mathrm{e}^{C_1\tau}\exp\left(\int_{\tau_0}^{\tau} s(t)\,\mathrm{sgn}(-V)\,\mathrm{d}t\right) \tag{18}$$

where we choose $C_1 = 1$ in the expansion phase and $C_1 = -1$ in the contraction phase of the volume function (if $V = 0$, $\ln(\mathrm{Vol}\,M^3)(\tau) \propto \mathrm{e}^{\pm\tau}$). Finally, for any model which describes the evolution of the volume function

$$\ln(\mathrm{Vol}\,M^3)(\tau) \propto \mathrm{e}^{\tau(1+\langle s\rangle)} \quad \overset{\tau\to\infty}{\longrightarrow} \quad \mathrm{e}^{\tau}\mathrm{e}^{\langle s\rangle} \propto \mathrm{e}^{\tau} \tag{19}$$

where

$$\langle s\rangle = \frac{1}{\tau}\int_{\tau_0}^{\tau} s(t)\,\mathrm{sgn}(-V)\,\mathrm{d}t$$

in which we assume that the average value of $s(\tau)$ on the interval (τ, τ_0) exists as $\tau \to -\infty$, and is finite. Equation (19) immediately yields the result that in a neighbourhood of the initial singularity ($\tau \to -\infty$) the volume function changes exactly as in Kasner's models. The second observation is as follows. the volume function does not oscillate around the equilibrium positions $\ln(\mathrm{Vol}\,M^3)(\tau) \equiv 0$ but oscillates around Kasner's solution. In other words, Kasner's solution plays a role analogous to that of the equilibrium positions in the small-oscillation approximation. From equation (18) one can obtain the following relations between a natural parameter s defined along the geodesics (Maupertuis time) and the volume function $\ln(\mathrm{Vol}\,M^3)(\tau)$:

$$s = \frac{\mathrm{d}}{\mathrm{d}t}\ln(\mathrm{Vol}\,M^3)(\tau) \qquad \text{for} \quad V < 0$$

$$s = -\frac{\mathrm{d}}{\mathrm{d}t}\ln(\mathrm{Vol}\,M^3)(\tau) \qquad \text{for} \quad V > 0.$$

The zero value of the parameter s corresponds to the moment $\tau = \tau_0$. The relations between the parameter s, the function F and the scalar expansion function

$$\Theta \equiv \frac{\mathrm{d}}{\mathrm{d}t} \ln(\mathrm{Vol}\, M^3)(t)$$

are as follows:

$$s(\tau) = \pm 6[(\mathrm{Vol}\, M^3)(\tau)]^{1/3} F(\tau)$$

$$s(\tau) = \pm[(\mathrm{Vol}\, M^3)(\tau)]\Theta(t(\tau))$$

where the plus and minus signs correspond to $V < 0$ and $V > 0$, respectively. From the above, we can conclude that 'near the singularity' is equivalent to $s \gg (\mathrm{Vol}\, M^3)^{1/3}$, i.e. $s \gg 0$. Equation (19) implies that the characteristic time after which $(\mathrm{Vol}\, M^3)(\tau)$ grows e-times, i.e. $(\mathrm{Vol}\, M^3)(\tau) \propto \mathrm{e}^{\tau/\tau_{\mathrm{char}}}$ has the following form:

$$\tau_{\mathrm{char}} = (1 + \langle s \rangle)^{-1}. \tag{20}$$

This characteristic time is finite if the average $\langle s \rangle$ exists.

4. Analysis of the integrability of the B(IX) model

There are several definitions of integrability. Generally, integrability means that the system under consideration possesses a sufficiently large number of first integrals. To be more precise, a Hamiltonian system with n degrees of freedom is integrable if it possesses n functionaly independent first integrals which are in involution (or which form a solvable Lie algebra). It is necessary to specify the class of functions that contains these first integrals as well as to define the domain of their definition. Let us note here that there are examples of Hamiltonian systems that possess a first integral of class C^α but do not possess an integral of class C^β with $\beta > \alpha$, for $\alpha, \beta = 1, 2, \ldots, \infty, \omega$ (see [19]).

It is also well known that every system of n differential autonomous equations is locally integrable—in a neighbourhood of every non-singular point (where the right-hand sides do not vanish) it possesses $n - 1$ first integrals. Thus, non-trivial problems are non-local or concern the existence of integrals in a neighbourhood of equilibrium points. It is very difficult to prove the integrability or otherwise of a given set of differential equations. One way to simplify the problem is to restrict the class of functions where we look for integrals.

As an illustration of this approach, let us consider the Birkhoff integrability (see [20]) of the Bianchi class A system in the form (12). This system belongs to the wide class of Hamiltonian systems in $\mathbb{R}^{2n}$ equipped with the standard symplectic structure and is given by the following Hamiltonian function:

$$\mathcal{H} = \frac{1}{2}(p, p) + \sum_{m \in \mathcal{M}} v_m \exp\langle c_m, q \rangle \tag{21}$$

where

$$(p, p) = \sum_{i,j=1}^{n} a^{ij} p_i p_j \qquad \langle c_m, q \rangle = \sum_{i=1}^{n} c_{m_i} q^i \quad m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$$

(a^{ij}), v_m and c_m are constant; $\mathcal{M}$ is a finite subset of $\mathbb{Z}^n$:

$$\mathcal{M} = \{m \in \mathbb{Z}^n \mid v_m \neq 0\}.$$

We look for integrals that are polynomials with respect to p, i.e.

$$f(q, p) = \sum_{k \in \mathcal{N}_l} f_k p^k$$

where

$$\mathcal{N}_l = \{m \in \mathbb{Z}_+^n \mid |m| \leqslant l\} \quad |m| = \sum_{i=1}^{n} m_i \qquad p^m = p_1^{m_1} \dots p_n^{m_n} \quad m \in \mathbb{Z}_+^n$$

and the coefficients f_k have the form of infinite series of exponents:

$$f_k = \sum_{m \in \mathbb{Z}^n} f_m^{(k)} \exp(c_m^{(k)}, q).$$

Here $\mathbb{Z}_+$ denotes non-negative integers. We say that the system (21) is Birkhoff integrable if it possesses n independent integrals of the prescribed form (see the note by Ziglin [21] about modification of the original definition of Kozlov). We order the elements of $\mathcal{M}$ according to lexicographic order and denote its maximal element by α and the maximal element of M that is not colinear with α by β. Then, according to [22, theorem 3] if

$$k(\alpha, \alpha) + (\alpha, \beta) \neq 0 \qquad \text{for all} \quad k \in \mathbb{Z}_+ \tag{22}$$

then the Hamiltonian system (21) is not integrable in the Birkhoff sense. We immediately have the following theorem.

Theorem 1. A generic case of the Bianchi class A system given by the Hamiltonian function (12) with $n_i \neq 0$ for $i = 1, 2, 3$ is not integrable in the Birkhoff sense.

Proof. For the Hamiltonian (12) we have

$$\mathcal{M} = \{(1, 1, 0), (1, 0, 1), (1, 0, 0), (0, 1, 1), (0, 1, 0), (0, 0, 1)\}$$

and thus $\alpha = (1, 1, 0)$ and $\beta = (1, 0, 1)$. The metric (a^{ij}) has the form

$$(a^{ij}) = 2 \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

and thus we have

$$k(\alpha, \alpha) + (\alpha, \beta) = 4 \neq 0$$

and this completes the proof. □

Let us remark that in the case where one of the n_i is equal to zero then $k(\alpha, \alpha) + (\alpha, \beta) = 0$ for all k. In such a case the system has one additional integral, namely p_i.

5. Reduction and non-integrability of the B(IX) model

The results obtained in section 4 are weak. There are two reasons for this. First, we asked about the *complete* integrability of the system. However, the system under investigation can only have *one additional integral*. The most important reason is the fact that we have formulated our question for the system defined on the whole of $\mathbb{R}^6$, although we are interested only in the system on a five-dimensional manifold defined by the level $\mathcal{H} = 0$. One can imagine a system that is not globally integrable although it possesses a single first integral on one prescribed energy surface. In fact, we can restrict our system to an energy surface, e.g., $\mathcal{H} = 0$, because it is invariant with respect to the flow generated by $\mathcal{H}$. Such a reduced system does not possess an *a priori* known first integral. However, let us assume that such integral F exits or can be found. Then, if it is defined on the whole phase space we can ask about invariant properties of surfaces of its constant value $F =$ constant. It turns out that such surfaces are in general *not invariant* with respect to the flow generated by $\mathcal{H}$; however, their parts which intersect the surface $\mathcal{H} = 0$ are invariant. An example of a system with such an integral is a heavy top in the Goryatshev–Tshaplygin case when an additional integral exists only on the zero level of the area first integral [10].

In this section, we want to study the B(IX) Hamiltonian system only on the level $\mathcal{H} = 0$.

When investigating a dynamical system we usually try to lower its dimension by making use of its first integrals and symmetries. For the Hamiltonian system (3), we know only one first integral, the Hamiltonian. Thus, using it, we can potentially reduce the dimension of the system by one; however, we lose the polynomial form of the system, and, moreover, the reduced Hamiltonian system thus obtained is not autonomous. We do not want this side effect of reduction, because it excludes the possibility of applying algebraic tools for the study of non-integrability.

In this section, we show how to reduce the dimension of the phase space by two and preserve the polynomial form of the vector field under consideration. We can achieve this although we lose the canonical Hamiltonian form of the system. In what follows, we consider the case of B(IX) ($n_1 = n_2 = n_3 = 1$). First, we transform the Hamiltonian vector field corresponding to the Hamiltonian (3) to a homogeneous polynomial form of second degree. To this end let us set

$$y_i = q^i \qquad z_i = \frac{\dot{q}^i}{q^i} \qquad i = 1, 2, 3 \tag{23}$$

then the equation of motion will have the form

$$\dot{y}_i = y_i z_i \qquad \dot{z}_i = (y_j - y_k)^2 - y_i^2 \qquad \{i, j, k\} \in S_3. \tag{24}$$

This system has a first integral corresponding to the Hamiltonian (3). It has the form

$$H = z_1 z_2 + z_1 z_3 + z_2 z_3 - y_1^2 + 2y_1 y_2 - y_2^2 + 2y_1 y_3 + 2y_2 y_3 - y_3^2. \tag{25}$$

We make the following change of variables:

$$w_1 = y_1 + y_2 \qquad w_2 = y_1 - y_2 \qquad w_3 = y_3 \tag{26}$$

and we leave z_i unchanged. In terms of the new variables the system (24) has the form

$$\begin{aligned}
\dot{w}_1 &= \tfrac{1}{2} z_1 (w_1 + w_2) + \tfrac{1}{2} z_2 (w_1 - w_2) \\
\dot{w}_2 &= \tfrac{1}{2} z_1 (w_1 + w_2) - \tfrac{1}{2} z_2 (w_1 - w_2) \\
\dot{w}_3 &= z_3 w_3 \\
\dot{z}_1 &= (w_3 - w_1)(w_2 + w_3) \\
\dot{z}_2 &= (w_3 - w_2)(w_3 - w_1) \\
\dot{z}_3 &= (w_3 + w_2)(w_2 - w_3)
\end{aligned} \tag{27}$$

and the first integral (25) is transformed to the following form:

$$H = z_1 z_2 + z_1 z_3 + z_2 z_3 - w_2^2 + 2w_1 w_3 - w_3^2. \tag{28}$$

Now, we introduce new variables

$$u_1 = \frac{z_1}{w_3} \qquad u_2 = \frac{z_2}{w_3} \qquad u_3 = \frac{z_3}{w_3} \qquad u_4 = \frac{w_2}{w_3} \qquad u_5 = \frac{w_1}{w_3} \qquad u_6 = w_3. \tag{29}$$

After this transformation we obtain the following system:

$$
\begin{aligned}
\dot{u}_1 &= u_6[(1+u_4)(1-u_5) - u_1u_3] \\
\dot{u}_2 &= u_6[(1-u_4)(1-u_5) - u_2u_3] \\
\dot{u}_3 &= u_6(u_4^2 - u_3^2 - 1) \\
\dot{u}_4 &= \tfrac{1}{2}u_6[u_4(u_1+u_2-2u_3) + u_5(u_1-u_2)] \\
\dot{u}_5 &= \tfrac{1}{2}u_6[u_4(u_1-u_2) + u_5(u_1+u2-2u_3)u_5] \\
\dot{u}_6 &= u_3u_6^2
\end{aligned}
\tag{30}
$$

with the first integral

$$H = u_6^2(u_1u_2 + u_1u_3 + u_2u_3 - u_4^2 + 2u_5 - 1). \tag{31}$$

Now, we make use of the fact that the B(IX) model is considered only on the level $H = 0$. From equation $H = 0$, we find u_5 as a function of (u_1, u_2, u_3, u_4):

$$u_5 = \tfrac{1}{2}(1 + u_4^2 - u_1u_2 - u_1u_3 - u_2u_3).$$

Thus we can eliminate this variable from the right-hand sides of (30). Moreover, if we change the independent variable according to the rule

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{u_6}{2}\frac{\mathrm{d}}{\mathrm{d}\bar{s}}$$

(note that $u_6 > 0$) then the first four equations in (30) are separated from the last two. Thus, we finally obtain the following close system describing the dynamic of the B(IX) model:

$$
\begin{aligned}
\dot{u}_1 &= (1+u_4)[1 + u_1u_2 + u_3(u_1+u_2) - u_4^2] - 2u_1u_3 \\
\dot{u}_2 &= (1-u_4)[1 + u_1u_2 + u_3(u_1+u_2) - u_4^2] - 2u_2u_3 \\
\dot{u}_3 &= 2(u_4^2 - u_3^2 - 1) \\
\dot{u}_4 &= u_4(u_1+u_2-2u_3) + \tfrac{1}{2}(u_1-u_2)[1 - u_1u_2 - u_3(u_1+u_2) + u_4^2].
\end{aligned}
\tag{32}
$$

This system will be called the reduced B(IX) system. We consider this system in $\mathbb{C}^4$.

Theorem 2. The reduced B(IX) system does not have a non-trivial analytic first integral.

Our theorem will be a consequence of the following lemma.

Lemma 1. Consider a system of differential equations

$$\dot{x} = f(x), \quad f(x) = (f_1(x), \ldots, f_n(x)) \qquad x \in \mathbb{C}^n \tag{33}$$

with analytic right-hand sides, with

$$f(x) = Ax + \mathrm{O}(|x|^2) \tag{34}$$

where the matrix A has eigenvalues $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, n$. If the system possesses an analytical first integral F then there exist non-negative integers $i_1, \ldots, i_n$ such that

$$\sum_{k=1}^{n} i_k\lambda_k = 0 \qquad \sum_{k=1}^{n} i_k > 0. \tag{35}$$

Let us assume that an analytic first integral exists and that condition (35) is not satisfied. We represent the first integral in the following form:

$$F = \sum_{l=k}^{\infty} F_l \qquad F_k \neq 0 \quad k \geqslant 1$$

where F_l is of a homogenous form of degree l:

$$F_l = \sum_{i_1+\cdots+i_n=l} F^{(l)}_{i_1,\ldots,i_n} x_1^{i_1}\cdots x_n^{i_n} \qquad i_k \in \mathbb{Z}_+ \quad k = 1,\ldots,n. \tag{36}$$

From the equation

$$\sum_{j=1}^{n} f_j(x)\partial_j F = 0$$

we conclude that the form F_k is a first integral of the system $\dot{x} = Ax$, i.e.

$$\sum_{j=1}^{n} l_j(x)\partial_j F_k = 0 \qquad l_i(x) = \sum_{j=1}^{n} A_{ij}x_j. \tag{37}$$

If the matrix A is diagonalizable then we can assume that $l_i(x) = \lambda_i x_i$, and then equation (37) reads

$$\sum_{i_1+\cdots+i_n=k} F^{(k)}_{i_1,\ldots,i_n}\left[\sum_{l=1}^{n} i_l\lambda_l\right] x_1^{i_1}\cdots x_n^{i_n} = 0. \tag{38}$$

This equation implies that

$$F^{(k)}_{i_1,\ldots,i_n}\left[\sum_{l=1}^{n} i_l\lambda_l\right] = 0 \quad \text{for all } (i_1,\ldots,i_n) \in \mathbb{Z}_+^n \qquad \sum_{l=1}^{n} i_l = k.$$

Because $F_k \neq 0$, there exist indices $(i_1,\ldots,i_n)$ such that $F^{(k)}_{i_1,\ldots,i_n} \neq 0$ and that for such indices we have

$$\sum_{l=1}^{n} i_l\lambda_l = 0.$$

Contradiction of our assumption proves the lemma for the case of a diagonalizable matrix A. In the case of a non-diagonalizable matrix the proof is only technically more difficult (see [23] and especially [24] where this approach was introduced in the generalized form).

Let us remark that the above lemma is also true if we assume the first integral is a formal power series.

To prove our theorem let us note that for the reduced B(IX) model point $z = (-i, -i, i, 0)$ is an equilibrium point and the matrix of the linearized system is diagonalizable and possesses the eigenvalues $(-2i, -2i, -4i, -4i)$. For these eigenvalues condition (35) cannot be satisfied and this implies that the system does not have an analytic first integral. In fact, we prove more, namely, the the system does not have a first integral that can be expanded around the point z as a formal power series.

6. Conclusions

The particular integrable subclasses of the Bianchi models play an important role in the analysis of the dynamics of cosmological models. To illustrate this fact, let us consider the phase space of the solutions of Bianchi models in the Bogoyavlensky approach [1]. In the Bogoyavlensky method for investigating the corresponding dynamical systems, we glue the boundary Δ onto which the system prolongs almost everywhere to the phase space. The systems on the boundary Δ can be integrated and, in this way, we can study the basic properties of the trajectories near the singularity. From the existence of the monotonic function F, we obtain the result that in the generic situation ($\forall\, i\ n_i \neq 0$) the trajectories of the Bianchi class A models close up to the boundary Γ as $F \ll -1$. Now, the trajectories move along the corresponding ones lying on the boundary. All the trajectories are the separatrices of critical points. Finally the trajectories reach the neighbourhood of the critical points K (corresponding to the Kasner asymptotics of the space–time metric) and they begin to move along their separatrices. The corresponding space–time metric for the mixmaster models is the BKL approximation [16].

In this way the chaotic (and thus non-integrable) systems in the Bogoyavlensky approach can be well approximated by an integrable system. This feature of such a surprisingly good approximation has so far not been completely understood. In Bianchi models with chaos, we have the infinite series of Kasner epochs (it would be good if we had a precise and exact proof of this fact) and these models do not exactly admit the Kasner asymptotics.

The fact that the BKL approximation represents a typical state of the metric in a very early state of the evolution has a very simple interpretation. For $F = F_1 \ll -1$ let there be some distribution of initial conditions (e.g., homogeneous). Approaching the initial singularity this distribution for $F = F_2$, where $F_2 < F_1$, transforms to the corresponding one concentrated in the neighbourhood of the critical points. These critical points have the separatrices which move towards the physical region of the phase space. During the motion along such a separatrix the space–time metric is described by the BKL approximation. From the fact that $F \to -\infty$ near the singularity, we can conclude the existence of a fundamental property of the system, namely the property of concentration of the trajectories near the boundary Δ [25].

In this paper we also try to obtain some exact results concerning integrability of the Bianchi models. It is important to note that in the B(IX) model the phase space is restricted to only one energy level $\mathcal{H} = 0$. Because of this, statements concerning the non-integrability of this model in the whole phase space are of limited value. What we propose consists of reduction of the system by one degree of freedom. For this purpose we use the energy integral and the symmetry of the system—in appropriate non-canonical coordinates the B(IX) system is a homogeneous one of the second degree. The reduced B(IX) system is polynomial and this allows us to prove its non-integrabilty. There are still open questions, e.g., does the reduced B(IX) system possess a meromorphic first integral? Is it chaotic?

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